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Quantum processes: probability fluxes, transition probabilities in unit time and vacuum vibrations

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Abstract. Transition probabilities in unit time and probability fluxes are compared in studying the elementary quantum processes—the decay of a bound state under the action of time-varying and constant electric fields. It is shown that the difference between these quantities may be considerable, and so the use of transition probabilities W instead of probability fluxes Π, in calculating the particle fluxes, may lead to serious errors. The difference between Π and W is due to the fact that in the formulae for probability fluxes the interference is taken into account between the transition amplitudes whereas in the formulae for transition probabilities there are no interference terms. The quantity W represents the rate of change with time of the population of the energy levels relating partly to the real states and partly to the virtual ones. For this reason it cannot be directly measured in experiment. Attention is drawn to the concept of the vacuum background that is treated as a physical medium consisting of virtual particles and as a framework in which the real quantum events occur. The vacuum background is shown to be continuously distorted when a perturbation acts on a system. Because of this the viewpoint of an observer on the physical properties of real particles continuously varies with time. This fact is not taken into consideration in the conventional theory of quantum transitions based on using the notion of probability amplitude. As a result, the probability amplitudes lose their physical meaning. All the physical information on quantum dynamics of a system is contained in the mean values of physical quantities. The existence of considerable differences between the quantities W and Π permits one in principle to make a choice of the correct theory of quantum transitions on the basis of experimental data.

1. Introduction

The main quantity measured in experiments on particle scattering is the number of particles emerging from the interaction region and passing in unit time through a certain surface defined by the solid-angle element $d\Omega$, that is the particle flux referred to the solid angle $d\Omega$ (Blokhintsev 1961). If a particle is described by the wavefunction $\Psi(\boldsymbol{r}, t)$, then in accordance with the principles of quantum mechanics the probability flux is determined by the expressions

$$d\Pi = \boldsymbol{j}\, d\boldsymbol{S} \qquad \boldsymbol{j} = \frac{i}{2m}\Psi(\boldsymbol{r}, t)\frac{\overleftrightarrow{\partial}}{\partial \boldsymbol{r}}\Psi^*(\boldsymbol{r}, t) \tag{1}$$

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where $d\boldsymbol{S}$ is the vector of the surface element corresponding to the solid angle $d\Omega$ and $\boldsymbol{j}$ is the probability flux density of the particle with the mass m. The integration of the quantity $d\Pi$ over the whole solid angle yields the total probability flux

$$\Pi = \oint_S \boldsymbol{j}\, d\boldsymbol{S}. \tag{2}$$

Here S is a closed surface enveloping the spatial region where the interaction occurs.

Introduce the transition probability in unit time

$$W = \frac{d}{dt} \sum_n |a_n(t)|^2 \tag{3}$$

where $a_n(t)$ are the expansion coefficients of the wavefunction $\Psi(\boldsymbol{r}, t)$ in the eigenfunctions of a certain operator (of the momentum operator, for example). In conventional scattering theory the quantities (3) are used as the probability fluxes. But even a brief inspection of the formulae (2) and (3) reveals that there is a significant difference between the quantities Π and W (see Oleinik 1985b, 1987b). Indeed, the probability flux (2) depends on the position in space of the surface S crossed by the registered particles whereas the transition probability in unit time (3), often called the rate of the process, is independent of the coordinates.

To clarify the main differences between Π and W, we dwell upon a specific quantum process. Consider a particle tunnelling out of the potential well by an applied field acting during the time interval $(0, T)$. Let $\{\varphi_h(\boldsymbol{r}, t)\}$ be the complete orthonormalised set of eigenfunctions of the energy operator of a particle in the potential well including both the continuous spectrum states (denoted hereafter by $\varphi_p(\boldsymbol{r}, t)$, $\boldsymbol{p}$ being the momentum) and the discrete spectrum states. Assume that $\varphi_0(\boldsymbol{r}, t)$ is the bound-state wavefunction of the particle at the initial moment of time $t = 0$ and $\Psi(\boldsymbol{r}, t)$ is the result of time evolution of the state $\varphi_0(\boldsymbol{r}, t)$ under the influence of the applied field. Expanding the wavefunction $\Psi(\boldsymbol{r}, t)$ at $t \geqslant T$ in terms of the unperturbed functions $\varphi_n(\boldsymbol{r}, t)$,

$$\Psi(\boldsymbol{r}, t) = \sum_n a_n(t)\varphi_n(\boldsymbol{r}, t) \tag{4}$$

and substituting (4) into (1), we arrive at the following relationship for the probability flux

$$\begin{aligned} \Pi &= -\frac{d}{dt} \int_V d\boldsymbol{r} |\Psi(\boldsymbol{r}, t)|^2 \\ &= -\frac{d}{dt} \sum_{n,n'} a_{n'}^*(t) a_n(t) \int_V d\boldsymbol{r} \varphi_{n'}^*(\boldsymbol{r}, t) \varphi_n(\boldsymbol{r}, t) \end{aligned} \tag{5}$$

where V is the volume of the region containing the potential well and bounded by the surface S. When deriving (5) we have made use of the Gauss theorem and the continuity equation.

Within the conventional theory the tunnelling transition of a particle out of the potential well is described in terms of the probability ionisation in unit time (the ionisation rate)

$$W = \frac{d}{dt} \int d\boldsymbol{p} |a_p(t)|^2 \tag{6}$$

where the quantity

$$a_p(t) = \int \mathrm{d}\boldsymbol{r}\varphi_p^*(\boldsymbol{r}, t)\Psi(\boldsymbol{r}, t) \tag{7}$$

represents the amplitude of the particle transition from the bound state φ_0 to the continuous spectrum state φ_p within the time interval $(0, t)$.

As is seen from the comparison of formulae (5) and (6), the main differences between them are as follows: (i) in (5) the summation is implied over all the states of continuous and discrete spectra whereas in (6) only the continuous spectrum states are taken into consideration; (ii) equation (5) allows for the interference between the transition amplitudes, as distinct from (6) which does not involve any interference terms. It should also be noted that the flux Π vanishes at the limit of infinitely large volume ($V \to \infty$). Indeed, when $V \to \infty$ the interference terms in (5) disappear due to orthogonality of the wavefunctions φ_n and $\varphi_{n'}$ at $n' \neq n$. By virtue of conservation in time of the normalisation integral $\Sigma_n |a_n(t)|^2 = \text{constant}$ and so we get $\Pi = 0$ for an arbitrary value of t. This equality results directly from the spreading in space of the wavepacket describing the localised particle state at the initial moment of time.

As was mentioned above, it is the transition probabilities in unit time W that are used in conventional scattering theory as the probability fluxes Π of scattered particles. In view of the fact that the quantities Π and W differ from each other, the following questions arise. What is the physical nature of the difference between Π and W? Which quantities provide adequate information on quantum transitions? What is the influence of the vacuum vibrations on physical processes? It is with the detailed consideration of these questions that this paper is concerned.

In § 2 an analysis is made of the measurements carried out in experiments on particle scattering. The arguments presented are in favour of the view that it is the probability fluxes rather than the transition probabilities in unit time that are registered in such experiments.

In §§ 3 and 4 the differences between the quantities Π and W are studied in detail in problems on the bound state decay under the influence of the time-varying and constant electric fields (Oleinik 1987b). It is shown that the difference between them increases with decreasing duration of the electric pulse causing decay. When the constant electric field is weak enough, the bound-state decay disobeys the exponential law. The basic decay mechanism turns out to be not the tunnel transitions of the particle through the potential barrier leading to the exponential decay law but its quantum jumps to the continuous spectrum states (Oleinik and Arepjev 1984a, b).

In § 5 the vacuum problems of quantum theory are discussed (Oleinik 1984, 1985a, b, 1986a, 1987b, Oleinik and Arepjev 1983a, b, 1984b). Here the model of the vacuum background is proposed as a medium formed by the virtual particles and as a framework in which the real quantum processes occur (Oleinik 1986a). A striking feature of the quantum dynamics of real microsystems is the continuous deformation of vacuum background in the course of time evolution. This deformation inevitably leads to a continuous change, from the observer's viewpoint, in the physical properties of a quantum particle. Conventional scattering theory is based on the assumption that the vacuum background does not change with time. This assumption is violated in real physical systems and, as a result, the probability amplitudes of quantum transitions lose their physical meaning. All the physical information on quantum processes is contained in the average values of physical quantities. These conclusions follow of necessity from the principles upon which the quantum theory rests.

In § 6 the main content of the paper is summarised.

2. What quantities are measured in experiments on particle scattering?

We shall turn now to analysing the measurement process to which the particles are subjected while studying the quantum transitions. We do not aim to give an exhaustive explanation of this point and merely dwell upon the problems which may shed some light on the relationship between the probability fluxes and the transition probabilities in unit time.

Let us recall the reasoning that led to the expressions of type (3). One of the main parts of the measuring device being used in experiments on particle scattering is the analyser performing the spectral resolution of the wavefunction (Blokhintsev 1961, 1987). If the diffraction grating is taken as an analyser, then the wavepacket is separated into the waves with different values of momentum. This corresponds to the expansion of type (4) where $\varphi_n = \varphi_p(\boldsymbol{r}, t)$ are the eigenfunctions of the momentum operator. Confining ourselves to the one-dimensional case, let us divide the region in which the momentum varies into intervals with the width Δp and denote by p_i the momentum pertaining to the middle of the ith interval. If N is the total number of measurements then, according to the statistical interpretation of quantum mechanics, the number of measurements resulting in the values of momentum in the interval $(p_i-(\Delta p/2), p_i+(\Delta p/2))$ will be N_i where

$$N_i/N = |a_{p_i}(t)|^2 \Delta p \tag{8}$$

$\Sigma_i N_i = N$. Summing the expression (8) over all the intervals we find that the total number of the particles observed is proportional to $\int \mathrm{d}p|a_p(t)|^2$. From this one can directly obtain the relation (3) which up to a constant factor gives the number of particles with arbitrary values of momentum registered by a detector in unit time. In other words, as follows from our reasoning, the number of particles with momentum lying in the interval $(p-(\Delta p/2), p+(\Delta p/2))$ registered in unit time is given by

$$\Delta W = \frac{\Delta|a_p|^2}{\Delta t}\Delta p \tag{9}$$

where

$$\Delta|a_p|^2 = |a_p(t_2)|^2 - |a_p(t_1)|^2 \qquad \Delta t = t_2 - t_1.$$

There is an important point which is often missed in considering the measurement process. Suppose that the operator $\mathscr{L}$ related to a physical quantity L has the eigenfunctions $\varphi_1, \varphi_2, \ldots$ corresponding to the eigenvalues $L_1, L_2, \ldots$ and prior to the measurement process the system in question was in a pure state described by the wavefunction Ψ. This may be represented in the form of a spectral resolution (4). As is claimed in the standard courses on quantum mechanics, if the measurement carried out on the system leads to the eigenvalue L_m of the quantity L, then after the measurement the system will belong to a new pure ensemble described by the wavefunction φ_m, that is, as a result of the measurement, a reduction of the wavepacket takes place:

$$\sum_n a_n\varphi_n \to \varphi_m. \tag{10}$$

The possibility of the reduction (10) does not give rise to doubt if the eigenvalue L_m belongs to the discrete spectrum. But if L_m corresponds to the continuous spectrum

(for instance, $L_m = p$, p is the momentum) such a reduction is not possible; otherwise, after measurement the system would become unobservable because the state would be in the form of a monochromatic wave, which cannot be experimentally registered. The wavefunction may only be reduced to a wavepacket of the type

$$\psi(x, t) = \int_{p-(\Delta p/2)}^{p+(\Delta p/2)} \mathrm{d}p' a_{p'}(t)\varphi_{p'}(x, t) \tag{11}$$

$$\varphi_p(x, t) = (2\pi)^{-1/2} \exp\left(\mathrm{i}px - \mathrm{i}\frac{p^2 t}{2m}\right)$$

the width Δx of the wavepacket and the magnitude of the momentum dispersion Δp being connected by the uncertainty relation $\Delta p \Delta x \geq \hbar/2$.

It follows from what has been said above that after measurement the particles whose momenta are within the interval $(p-(\Delta p/2), p+(\Delta p/2))$ should be described by the wavepacket $\psi(x, t)$ (11) rather than by the monochromatic wave $\varphi_p(x, t)$. The probability flux for the wavepacket (11) is expressed by the equation

$$\mathrm{d}\Pi(x) = \frac{\mathrm{i}}{2m} \psi(x, t) \overleftrightarrow{\frac{\partial}{\partial x}} \psi^*(x, t) \tag{12}$$

which is different from (9) (in the one-dimensional case the probability flux density coincides with the probability flux).

On the other hand, making use of the expression (11), we find that the change in unit time of the probability of the particle having in whole space the momenta within the interval mentioned above is

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{+\infty} |\psi(x, t)|^2 \, \mathrm{d}x = \int_{p-(\Delta p/2)}^{p+(\Delta p/2)} \frac{\mathrm{d}|a_p(t)|^2}{\mathrm{d}t} \mathrm{d}p.$$

This expression coincides, as it should, with (9) at $\Delta p \to 0$, $\Delta t \to 0$. To avoid possible misunderstanding, it should be noted that if the initial wavepacket $\Psi(x, t) = \int_{-\infty}^{+\infty} \mathrm{d}p' \, a_{p'}(t)\varphi_{p'}(x, t)$ obeys the time-dependent Schrödinger equation, the reduced wavepacket (11), generally speaking, does not.

Thus, the quantities (9) and (12) essentially differ from one another in their physical meaning: quantities of the type $\mathrm{d}W$ yield the changes in unit time of the number of particles in whole space on the condition that their momenta lie within the interval of the width Δp, whereas quantities of the type $\mathrm{d}\Pi$ determine only the number of particles that cross a certain surface with the same limitation on the magnitude of the momentum. The position in space, the form and dimensions of the surface are determined by the conditions of experiment (in particular, by the relative arrangement of analyser and detector used when performing the measurements). The quantities $\mathrm{d}W$ do not contain any information on the surface which is crossed by the registered particles; in other words, these quantities do not allow for certain things that are essential in conducting the real measurements and which have an influence on their results.

The difference between the probability fluxes and the transition probabilities in unit time can be illustrated by the following example. Let the registered particle be described by the wavefunction

$$\psi_p(x, t) = \int_{-\infty}^{+\infty} \mathrm{d}p' \, a_{p'}\varphi_{p'}(x, t) \qquad a_{p'} = c \exp\left(-\frac{(p'-p)^2}{b^2}\right)$$

where $c=(2/\pi)^{1/4}b^{-1/2}$ is the normalisation constant, b and p are constants and the function $\varphi_p(x, t)$ is defined in (11). Then the probability flux and the transition probability in unit time are expressed by the formulae

$$\begin{aligned}\Pi(x, t) &= \frac{\mathrm{i}}{2m}\psi_p(x, t)\overleftrightarrow{\frac{\partial}{\partial x}}\psi_p^*(x, t)\\ &= (2\pi)^{-1/2}\frac{b^2}{m}\left(\frac{p}{b}+\frac{xb}{2}\frac{tb^2}{2m}\right)\left[1+\left(\frac{tb^2}{2m}\right)^2\right]^{-3/2}\\ &\quad\times\exp\left[-\frac{b^2}{2}\left(x-\frac{p}{m}t\right)^2\left(1+\left(\frac{tb^2}{2m}\right)^2\right)^{-1}\right]\\ W &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{+\infty}\mathrm{d}p|a_p|^2=0.\end{aligned} \tag{13}$$

Note the equality $\Pi(x, t)=0$ at $x\to\pm\infty$ and at an arbitrary value of t. As is seen from (13), the difference between the quantities Π and W may be considerable. From this it follows, in particular, that the substitution of the probability fluxes by the transition probabilities in unit time in formulae defining the cross sections of quantum processes may lead to serious errors.

The main difference between the probability flux and the transition probability in unit time is that the former has a 'registration' in a definite region of space (in the one-dimensional case, for instance, the magnitude of the flux depends on coordinates; see (13)) and the latter refers to the whole space and does not depend on coordinates. The quantities Π and W are, respectively, the spacetime characteristic of a process and the momentum-energy one (Bykov and Zadernovsky 1981, Bykov and Shepelev 1986). In accordance with the Bohr complementarity principle (Bohr 1971) the space-time and momentum-energy dynamical variables represent the two complementary and mutually exclusive groups of variables which cannot manifest themselves simultaneously in one and the same quantum ensemble.

In the next two sections the differences between the quantities Π and W are analysed in detail using the examples of elementary quantum processes.

3. Bound-state decay in a time-varying electric field

Let us consider the decay of the bound state of a particle localised in the potential well

$$U_0(z)=-\frac{\kappa_0}{m}\delta(z)\qquad\kappa_0>0 \tag{14}$$

under the influence of the time-varying electric field $\mathscr{E}(t)$ (Oleinik 1986a, 1987b). The potential energy of the particle is given by

$$U_1(z, t)=-e\mathscr{E}(t)z.$$

The only energy level $E_0=-\kappa_0^2/2m$ corresponding to the bound state in the potential well (14) is described by

$$\varphi_{E_0}(z)=\kappa_0^{1/2}[\theta(-z)\,\mathrm{e}^{\kappa_0 z}+\theta(z)\,\mathrm{e}^{-\kappa_0 z}]. \tag{15}$$

Denote by $\varphi_{En}(z)$ $(n = s, a)$ the wavefunctions relating to the continuous energy spectrum:

$$\varphi_{Es}(z) = a_{Es}\{k\cos kz - \kappa_0 \sin kz[\theta(z) - \theta(-z)]\}$$
$$\varphi_{Ea}(z) = a_{Ea}\sin kz \qquad k = (2mE)^{1/2} \qquad E \geqslant 0 \tag{16}$$
$$a_{Ea}^2 = \frac{m}{\pi k} \qquad a_{Es}^2 = m(k^2 + \kappa_0^2)^{-1}\pi^{-1}k^{-1}.$$

The wavefunctions (15) and (16) satisfy the following conditions of completeness and orthonormalisation:

$$\sum_{n=a,s}\int_0^\infty dE\,\varphi_{En}^*(z_1)\varphi_{En}(z_2) + \varphi_{E_0}(z_1)\varphi_{E_0}(z_2) = \delta(z_1 - z_2)$$
$$\int dz\,\varphi_{En}^*(z)\varphi_{E'n'}(z) = \delta(E - E')\delta_{nn'} \tag{17}$$
$$\int dz\,\varphi_{En}^*(z)\varphi_{E_0}(z) = 0 \qquad \int dz|\varphi_{E_0}(z)|^2 = 1.$$

Consider the bound-state decay by making use of the Drukarjev method (Drukarjev 1951). Let us suppose that prior to the moment of time $t = 0$ the system is in the bound state φ_{E_0} and at $t = 0$ the electric field $\mathscr{E}(t)$ causing the decay is switched on. Denote by $\Psi_{E_0}(z, t)$ the solution of the time-dependent Schrödinger equation in the field $U_0(z) + U_1(z, t)$ obeying the initial condition

$$\Psi_{E_0}(z, 0) = \varphi_{E_0}(z). \tag{18}$$

By a decay we mean here the time evolution of the system in an external field; that is, the quantum transition $\varphi_{E_0}(z) \to \Psi_{E_0}(z, t)$.

According to the conventional theory of quantum transitions, the probability amplitude of the transition of the system from the state φ_{E_0} to the state φ_{En} by the moment $t (t > 0)$ is given by

$$M_{E_0 \to En}(t) = \int_{-\infty}^{+\infty} dz\varphi_{En}^*(z, t)\Psi_{E_0}(z, t) \qquad \varphi_{En}(z, t) = \varphi_{En}(z)\,e^{-iEt}. \tag{19}$$

The total transition probability in unit time from the bound state to the continuous spectrum states (the ionisation rate of the potential well) is of the form

$$W(t) = \frac{d}{dt}\sum_{n=a,s}\int_0^\infty dE|M_{E_0 \to E_n}(t)|^2. \tag{20}$$

The probability flux density $j(z, t)$ and the total probability flux $\Pi(z, t)$ from the region $(-z, +z)$ are determined by the expressions

$$j(z, t) = \frac{i}{2m}\Psi_{E_0}(z, t)\frac{\overleftrightarrow{\partial}}{\partial z}\Psi_{E_0}^*(z, t) \tag{21}$$

$$\Pi(z, t) = \int_{-z}^{+z} dz'\,\mathrm{div}\,j(z', t) = j(z, t) - j(-z, t) \tag{22}$$

where $j = (0, 0, j)$. Our aim is to compare in detail the quantities $\Pi(z, t)$ and $W(t)$ for the electric field $\mathscr{E}(t)$. Assuming the field $\mathscr{E}(t)$ to be a small perturbation, we confine ourselves to the second-order terms.

With the required accuracy the wavefunction $\Psi_{E_0}(z, t)$ may be represented as follows:

$$\Psi_{E_0}(z, t) = \varphi_{E_0}(z, t) + \Psi^{(1)}_{E_0}(z, t) + \Psi^{(2)}_{E_0}(z, t)$$
$$\Psi^{(1)}_{E_0}(z, t) = -\mathrm{i} \sum_{En} \varphi_{En}(z, t) \int_0^t \mathrm{d}t_1 \, \mathcal{M}_{En, E_0}(t_1) \tag{23}$$
$$\Psi^{(2)}_{E_0}(z, t) = -\sum_{En} \varphi_{En}(z, t) \sum_{E'n'} \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \, \mathcal{M}_{En, E'n'}(t_1) \mathcal{M}_{E'n', E_0}(t_2)$$

$$\mathcal{M}_{En, E'n'}(t) = \int_{-\infty}^{+\infty} \mathrm{d}z \, \varphi^*_{En}(z, t) U_1(z, t) \varphi_{E'n'}(z, t). \tag{24}$$

Here the symbol Σ_{En} means the integration over continuous spectrum states and the summation over discrete spectrum ones. Substituting the expression (23) into (19) and retaining only terms of first order in U_1, we arrive at

$$M_{E_0 \to En}(t) = -\mathrm{i} \int_0^t \mathrm{d}t_1 \, \mathcal{M}_{En, E_0}(t_1) \delta_{na}. \tag{25}$$

This allows for the fact that the interaction Hamiltonian $U_1(z, t)$ is an odd function in z and the function $\varphi_{E_0}(z)$ is an even one.

Note that the component of the probability flux density (21), equal to

$$\frac{\mathrm{i}}{2m} \varphi_{E_0}(z, t) \frac{\overleftrightarrow{\partial}}{\partial z} \Psi^{*(1)}_{E_0}(z, t) + \mathrm{CC}$$

is an even function in z, so it makes no contribution to the total flux (22). For simplicity, we calculate the flux $\Pi(z, t)$ at large values of $|z|$ such that

$$\kappa_0 |z| \gg 1. \tag{26}$$

In this case the component of the probability flux density, equal to

$$\frac{\mathrm{i}}{2m} \varphi_{E_0}(z, t) \frac{\overleftrightarrow{\partial}}{\partial z} \Psi^{*(2)}_{E_0}(z, t) + \mathrm{CC}$$

will be exponentially small; therefore it may be neglected. As a result, we obtain the following formula for the total probability flux:

$$\Pi(z, t) = \frac{\mathrm{i}}{m} \Psi^{(1)}_{E_0}(z, \mathrm{t}) \frac{\overleftrightarrow{\partial}}{\partial z} \Psi^{*(1)}_{E_0}(z, t). \tag{27}$$

Making use of (15), (16) and (24), we calculate the matrix element:

$$\mathcal{M}_{E_a, E_0}(t) = -4 a_{Ea} \kappa_0^{3/2} k (k^2 + \kappa_0^2)^{-2} e \mathscr{E}(t) \exp[\mathrm{i}(E - E_0)t]. \tag{28}$$

Further, from the relationships (20), (23), (25) and (28) we can infer

$$W(t) = 32\pi^{-1} \kappa_0^3 \, \mathrm{e} \, \mathscr{E}(t) \times \int_0^\infty \mathrm{d}k \frac{k^2}{(k^2 + \kappa_0^2)^4} \int_0^t \mathrm{d}t_1 e \mathscr{E}(t_1) \cos\left(\frac{k^2 + \kappa_0^2}{2m} (t - t_1) \right)$$
$$\psi^{(1)}_{E_0}(z, t) = \mathrm{i} \, 4\pi^{-1} \kappa_0^{3/2} \, \mathrm{e}^{-\mathrm{i}E_0 t} \times \int_0^\infty \mathrm{d}k \frac{k \sin kz}{(k^2 + \kappa_0^2)^2} \int_0^t \mathrm{d}t_1 e \mathscr{E}(t_1) \exp\left(-\mathrm{i} \frac{k^2 + \kappa_0^2}{2m} (t - t_1) \right). \tag{29}$$

As the electric field we take

$$\mathscr{E}(t) = \int \mathrm{d}\omega\, \mathscr{E}_\omega \cos \omega t \qquad \mathscr{E}_\omega = \pi^{-1}\Gamma \mathscr{E}_0[(\omega-\omega_0)^2+\Gamma^2]^{-1}$$

that is

$$\mathscr{E}(t) = \mathscr{E}_0 \cos \omega_0 t\, \mathrm{e}^{-\Gamma t} \qquad t \geqslant 0. \tag{30}$$

The quantity Γ^{-1} may be interpreted as the duration of a pulse of electric field.

After performing the integration over t_1 in (29), we have

$$\begin{aligned} W(t) &= 16\pi^{-1}\kappa_0^3(e\mathscr{E}_0)^2 m \cos \omega_0 t\, \mathrm{e}^{-\Gamma t} \\ &\quad \times \sum_{\sigma=\pm 1} [\mathrm{i}(\mathscr{K}_\sigma(t)\, \mathrm{e}^{\mathrm{i}E_0 t} - \mathscr{K}_\sigma(0)\, \mathrm{e}^{-\mathrm{i}\sigma\omega_0 t - \Gamma t}) + \mathrm{CC}] \\ \Psi^{(1)}_{E_0}(z, t) &= -2\pi^{-1}\kappa_0^{3/2} e\mathscr{E}_0 m \sum_{\sigma=\pm 1} (\mathscr{L}_\sigma(t, z) - \mathscr{L}_\sigma(0, z)\, \mathrm{e}^{\mathrm{i}tf_{2\sigma}/2m}). \end{aligned} \tag{31}$$

Above we have used the notation

$$\begin{aligned} \mathscr{K}_\sigma(t) &= \int_0^\infty \mathrm{d}k\, k^2(k^2+f_1)^{-4}(k^2+f_{2\sigma})^{-1} \exp\left(-\mathrm{i}\frac{k^2}{2m}t\right) \\ \mathscr{L}_\sigma(t, z) &= \frac{\partial}{\partial f_1}\frac{\partial}{\partial z}\int_{-\infty}^{+\infty} \mathrm{d}k(k^2+f_1)^{-1}(k^2+f_{2\sigma})^{-1} \exp\left(\mathrm{i}kz - \mathrm{i}\frac{k^2}{2m}t\right) \\ f_1 &= \kappa_0^2 \qquad f_{2\sigma} = 2m(|E_0| - \sigma\omega_0 + \mathrm{i}\Gamma). \end{aligned} \tag{32}$$

For convenience the functions $\mathscr{K}_\sigma(t)$ and $\mathscr{L}_\sigma(t, z)$ may be expressed in terms of the following integrals:

$$Y(f) = \int_{-\infty}^{+\infty} \mathrm{d}k(k^2+f)^{-1} \exp\left(\mathrm{i}kz - \mathrm{i}\frac{k^2}{2m}t\right) \qquad \text{and} \qquad Y_0(f) = Y(f)|_{z=0} \tag{33}$$

which are connected with the probability function $\mathrm{erfc}(z)$ by

$$Y(f_n) = \left(\frac{\pi}{f_n}\right)^{1/2} \mathrm{e}^{\mathrm{i}tf_n/2m}[\mathrm{e}^{-z\sqrt{f_n}}\, \mathrm{erfc}(-a_n^{(-)}) + \mathrm{e}^{z\sqrt{f_n}}\, \mathrm{erfc}(a_n^{(+)})]$$

$$\mp a_n^{(\mp)} = \mp\left(z \mp \mathrm{i}\frac{t}{m}\sqrt{f_n}\right)\left(\frac{m}{2t}\right)^{1/2} \mathrm{e}^{-\mathrm{i}\pi/4} \tag{34a}$$

$$Y_0(f_n) = 2\left(\frac{\pi}{f_n}\right)^{1/2} \mathrm{e}^{\mathrm{i}tf_n/2m}\, \mathrm{erfc}\left(\sqrt{\frac{tf_n}{2m}}\, \mathrm{e}^{\mathrm{i}\pi/4}\right) \tag{34b}$$

$$\mathrm{erfc}(z) = \int_z^\infty \mathrm{d}x\, \mathrm{e}^{-x^2}.$$

For the functions $\mathscr{K}_\sigma(t)$ and $\mathscr{L}_\sigma(t, z)$ we have the following representations:

$$\begin{aligned} \mathscr{K}_\sigma(t) &= -\frac{1}{12}\frac{\partial^3}{\partial f_1^3}\{(f_1-f_{2\sigma})^{-1}[f_1\, Y_0(f_1) - f_{2\sigma}\, Y_0(f_{2\sigma})]\} \\ \mathscr{L}_\sigma(t, z) &= \frac{\partial}{\partial f_1}\frac{\partial}{\partial z}\{(f_{2\sigma}-f_1)^{-1}[\, Y(f_1) - Y(f_{2\sigma})]\} \end{aligned} \tag{35}$$

from which we can infer the equalities

$$\mathscr{K}_\sigma(0) = -\frac{\pi}{12}\frac{\partial^3}{\partial f_1^3}(f_1^{1/2}+f_{2\sigma}^{1/2})^{-1}$$
$$\mathscr{L}_\sigma(0,z) = \frac{\pi}{(f_{2\sigma}-f_1)^2}[-e^{-z\sqrt{f_1}}+e^{-z\sqrt{f_{2\sigma}}}]+\frac{\pi z f_1^{-1/2}}{2(f_{2\sigma}-f_1)}e^{-z\sqrt{f_1}}. \tag{36}$$

We confine ourselves to considering the region of parameters where

$$\Gamma \ll |\omega_0 - |E_0|| \ll |E_0| \tag{37}$$
$$\frac{t|f_n|}{2m} \gg 1 \qquad z \ll \frac{t}{m}|f_n|^{1/2} \qquad \frac{z^2 m}{2t} \ll 1 \qquad (n=1, 2\sigma).$$

For illustration the following numerical values of parameters are given: if

$$|E_0| = 0.5\ \text{eV} \qquad \Gamma = 10^{-4}|E_0| \qquad |\omega_0 - |E_0|| = 10^{-3}|E_0| \tag{38}$$
$$\Gamma t \geqslant 1 \qquad \kappa_0 z = 10^2$$

then

$$\frac{t|f_{21}|}{2m} \geqslant 10 \qquad \frac{tf_1}{2m} \geqslant 10^4 \qquad \frac{z^2 m}{2t} \leqslant \frac{1}{4} \qquad t \geqslant \Gamma^{-1} = 1.3\times 10^{-11}\ \text{s}. \tag{39}$$

Case 1. $\omega_0 > |E_0|$. In the region of parameters (37) the following inequalities are satisfied:

$$\operatorname{Re}(\pm a_n^{(\pm)}) > 0 \qquad \operatorname{Re}\left(\sqrt{\frac{tf_n}{2m}}\,e^{i\pi/4}\right) > 0 \qquad n = 1; 2, -1$$
$$\operatorname{Re}(\pm a_{2,1}^{(\pm)}) < 0 \qquad \operatorname{Re}\left(\sqrt{\frac{tf_{21}}{2m}}\,e^{i\pi/4}\right) < 0. \tag{40}$$

Taking into account the formula

$$\operatorname{erfc}(-z) = \sqrt{\pi} - \operatorname{erfc}(z) \qquad \operatorname{Re} z > 0$$

the asymptotic form of the probability function

$$\operatorname{erfc}(z) = e^{-z^2}\left(\frac{1}{2z}-\frac{1}{4z^3}+\frac{3}{8z^5}-\ldots\right) \qquad \operatorname{Re} z > 0, |z| \gg 1 \tag{41}$$

and the relations (34) and (35) as well, we obtain

$$\mathscr{K}_\sigma(t) = -\frac{\pi f_{21}^{1/2}}{(f_1-f_{21})^4}e^{it f_{21}/2m}\delta_{\sigma 1} + \frac{\sqrt{\pi}}{4}\left(\frac{it}{2m}\right)^{-3/2} f_1^{-4} f_{2\sigma}^{-1}. \tag{42}$$

It is obvious that $|\mathscr{K}_1(t)| \gg |\mathscr{K}_{-1}(t)|$, the first term on the right-hand side of the equality (42) being much greater than the second term (at $\Gamma t \leqslant 1$). Note that in (31) for $W(t)$ there are high-frequency terms of the type $\exp(\pm i2\omega_0 t)$. These terms are omitted and only the highest magnitude values are retained. The final result is given by

$$W(t) = 8\kappa_0^3 (e\mathscr{E}_0)^2 m |f_{21}|^{1/2} f_1^{-4} e^{-\Gamma t}$$
$$\times\left[e^{-\Gamma t} - \frac{1}{2\sqrt{\pi}}\left(\frac{t|f_{21}|}{2m}\right)^{-3/2}\cos\left(|E_0|-\omega_0)t+\frac{\pi}{4}\right)\right]. \tag{43}$$

In the same way we calculate the function $\mathscr{L}_\sigma(t, z)$ (35) defining the wavefunction $\Psi^{(1)}_{E_0}(z, t)$. Omitting the intermediate calculations, we write out the final expression:

$$\Psi^{(1)}_{E_0}(z, t) = 2\kappa_0^{3/2} e\mathscr{E}_0 m f_1^{-2} \times \left[\exp(\mathrm{i}A_1 - A_2) - \frac{z\sqrt{|f_{21}|}}{2\sqrt{\pi}} \left(\frac{t|f_{21}|}{2m} \right)^{-3/2} \exp\left(\mathrm{i}\frac{mz^2}{2t} + \mathrm{i}\frac{\pi}{4} \right) \right]. \tag{44}$$

Here the following notation is introduced:

$$\mathrm{i}\frac{tf_{21}}{2m} + z(f_{21})^{1/2} = \mathrm{i}A_1 - A_2$$

$$A_1 = (|E_0| - \omega_0)t + z[2m(\omega_0 - |E_0|)]^{1/2}$$

$$A_2 = \Gamma(t - mz|f_{21}|^{-1/2}).$$

The total probability flux is determined by

$$\Pi(z, t) = 8\kappa_0^3 (e\mathscr{E}_0)^2 m |f_{21}|^{1/2} f_1^{-4} \mathrm{e}^{-A_2} \times \left\{ \mathrm{e}^{-A_2} - \frac{1}{2\sqrt{\pi}} \left(\frac{t|f_{21}|}{2m} \right)^{-3/2} \left[\cos\left(A_1 - \frac{mz^2}{2t} + \frac{\pi}{4} \right) + z|f_{21}|^{1/2} \cos\left(A_1 - \frac{mz^2}{2t} - \frac{\pi}{4} \right) \right] \right\}. \tag{45}$$

As is seen from the comparison of (43) with (45), the marked difference between these formulae arises at

$$z|f_{21}|^{1/2} \geqslant 1 \qquad \Gamma mz|f_{21}|^{-1/2} \geqslant 1. \tag{46}$$

At the values of parameters indicated in (38) and (39) we have:

$$z|f_{21}|^{1/2} = \sqrt{10} \qquad \Gamma mz|f_{21}|^{-1/2} = \frac{1}{2\sqrt{10}}.$$

The formula (45) also remains approximately valid when, instead of the last two inequalities (37), the following inequalities are fulfilled:

$$z < \frac{t}{m}|f_{21}|^{1/2} \qquad |a^{(\pm)}_{2,1}| \gg 1.$$

Then the quantity $|A_2|$ may be of the order of 1 while $\Gamma t \gg 1$ (that is $\mathrm{e}^{-A_2} \sim 1$, $\mathrm{e}^{-\Gamma t} \ll 1$). In this case the difference between the quantities $W(t)$ and $\Pi(z, t)$ will be particularly substantial. Thus, with increasing Γ, that is with decreasing duration of the electric pulse, the difference between $W(t)$ and $\Pi(z, t)$ becomes greater.

Case 2. $\omega_0 < |E_0|$. Now in the region of parameters (37) the following inequalities are satisfied:

$$\mathrm{Re}(\pm a^{(\pm)}_n) > 0 \qquad \mathrm{Re}\left(\sqrt{\frac{tf_n}{2m}}\, \mathrm{e}^{\mathrm{i}\pi/4} \right) > 0 \qquad \text{at all } n.$$

From (34), (35) and (41) we infer

$$\mathscr{K}_\sigma(t) = \tfrac{1}{4}\pi^{1/2} \left(\frac{\mathrm{i}t}{2m} \right)^{-3/2} f_1^{-4} f_{2\sigma}^{-1}. \tag{47}$$

Retaining in (31) again only the highest magnitude terms, we arrive at the equality

$$W(t) = -8\kappa_0^3(e\mathscr{E}_0)^2 m f_1^{-4}\,\mathrm{e}^{-\Gamma t} \times\left[m\Gamma|f_{21}|^{-1/2}\,\mathrm{e}^{-\Gamma t} - \frac{1}{2\sqrt{\pi}}|f_{21}|^{1/2}\left(\frac{t|f_{21}|}{2m}\right)^{-3/2}\cos\left((|E_0|-\omega_0)t+\frac{\pi}{4}\right)\right]. \tag{48}$$

The analogous expression for the probability flux is of the form

$$\Pi(z,t) = -8\kappa_0^3(e\mathscr{E}_0)^2 m f_1^{-4}\,\mathrm{e}^{-B_2} \times\{m\Gamma|f_{21}|^{-1/2}\,\mathrm{e}^{-B_2} - \frac{1}{2\sqrt{\pi}}|f_{21}|^{1/2}\left(\frac{t|f_{21}|}{2m}\right)^{-3/2} \times\left[\cos\left(B_1-\frac{mz^2}{2t}+\frac{\pi}{4}\right)+z|f_{21}|^{1/2}\cos\left(B_1-\frac{mz^2}{2t}-\frac{\pi}{4}\right)\right]\}. \tag{49}$$

The following notation is used here:

$$\mathrm{i}\frac{tf_{21}}{2m} - z(f_{21})^{1/2} = \mathrm{i}B_1 - B_2$$

$$B_1 = (|E_0|-\omega_0)t - zm\Gamma|f_{21}|^{-1/2} \qquad B_2 = \Gamma t + z|f_{21}|^{1/2}.$$

From the formulae given above it follows that the marked difference between $W(t)$ and $\Pi(z,t)$ arises, as in case 1, when the conditions (46) are fulfilled. The first terms on the right-hand sides of (48) and (49) differ by the factor $\exp[-2z|f_{21}|^{1/2}]$ which can be very small (at the values of parameters (38) and (39) $\exp[-2z|f_{21}|^{1/2}] = 1.8\times10^{-3}$).

Thus, in a rather wide region of parameters of the problem the probability fluxes differ appreciably from the transition probabilities in unit time. However, there are also regions of parameters where the quantities under study do not practically differ from each other.

4. Bound-state decay in a constant electric field

In this section we consider the particle escaping from the delta function potential well (14) under the action of the constant electric field $\mathscr{E}$ (Oleinik 1987a). We follow the same statement of the physical problem as in the previous section but the electric field is taken into consideration precisely, without using perturbation theory (Oleinik and Arepjev 1984a).

Denote by $\phi_E(z)$ the energy eigenfunctions of the electron in the potential field

$$U(z) = -\frac{\kappa_0}{m}\delta(z) - e\mathscr{E}z \tag{50}$$

and by $\Psi_{E_0}(z,t)$ the solution of the Schrödinger equation in the field (50) satisfying the initial condition (18). The wavefunction $\Psi_{E_0}(z,t)$ may be expended in the functions $\phi_E(z)$:

$$\Psi_{E_0}(z,t) = \int_{-\infty}^{+\infty} \mathrm{d}E c_{EE_0}\phi_E(z)\,\mathrm{e}^{-\mathrm{i}Et}. \tag{51}$$

Here c_{EE_0} are the constant coefficients defined by the initial condition (18):

$$c_{EE_0} = \int_{-\infty}^{+\infty} \mathrm{d}z \phi_E^*(z) \varphi_{E_0}(z). \tag{52}$$

Making use of the condition of completeness (17), the formula (20) for the ionisation probability may be transformed as follows:

$$\begin{aligned} W(t) &= -\frac{\mathrm{d}}{\mathrm{d}t} |\mathcal{M}_{E_0}(t)|^2 \\ \mathcal{M}_{E_0}(t) &= \int_{-\infty}^{+\infty} \mathrm{d}z \, \varphi_{E_0}^*(z, t) \Psi_{E_0}(z, t). \end{aligned} \tag{53}$$

Substituting the representation (51) into the last equation, we obtain

$$\mathcal{M}_{E_0}(t) = \int_{-\infty}^{+\infty} \mathrm{d}E |c_{EE_0}|^2 \, \mathrm{e}^{-\mathrm{i}(E-E_0)t}. \tag{54}$$

The total probability flux can be worked out from (21), (22) and (51). The wavefunction $\phi_E(z)$ is determined by

$$\begin{aligned} &\phi_E(z) = \alpha_E \{\mathrm{Ai}(-\xi)\theta(-z) + [a_E \, \mathrm{Ai}(-\xi) + b_E \, \mathrm{Bi}(-\xi)]\theta(z)\} \\ &a_E = 1 - 2\pi\beta \, \mathrm{Ai}(-\xi_0) \, \mathrm{Bi}(-\xi_0) \qquad b_E = 2\pi\beta \, \mathrm{Ai}^2(-\xi_0) \\ &|\alpha_E|^2 = 2m\beta\kappa_0^{-1}(a_E^2 + b_E^2)^{-1} \\ &\xi = \kappa_0 \beta^{-1} z + \xi_0 \qquad \xi_0 = \beta^2 \frac{E}{|E_0|}. \end{aligned} \tag{55}$$

Here β is a dimensionless parameter defined by the formula $\beta = \kappa_0 (2me\mathscr{E}\hbar)^{-1/3}$ (in ordinary units); $\mathrm{Ai}(-\xi)$ and $\mathrm{Bi}(-\xi)$ are Airy functions.

The function $\mathcal{M}_{E_0}(t)$ (54) can be rewritten in the form

$$\mathcal{M}_{E_0}(t) = \int_{-\infty}^{+\infty} \mathrm{d}x \, \mathrm{e}^{-\mathrm{i}x\tau} |P(x)|^2 (a_x^2 + b_x^2)^{-1} \tag{56}$$

where $a_x = a_E|_{E=x|E_0|}$; $b_x = b_E|_{E=x|E_0|}$; the function $P(x)$ is determined by

$$c_{EE_0} = |E_0|^{-1/2} \left(\frac{\kappa_0}{2m\beta}\right)^{1/2} \alpha_E^* P\left(\frac{E}{|E_0|}\right). \tag{57}$$

Here and below the dimensionless time $\tau = |E_0|t/\hbar$ and coordinate $Z = \kappa_0 z/\hbar$ are introduced. In terms of these variables the ionisation probability in unit time and the probability flux density are of the form

$$\begin{aligned} &W(t) = \frac{|E_0|}{\hbar} \tilde{W}(\tau) \qquad j(z, t) = \frac{|E_0|}{\hbar} \tilde{j}(Z, \tau) \\ &\tilde{W}(\tau) = -\frac{\mathrm{d}}{\mathrm{d}\tau} |\mathcal{M}_{E_0}(t)|^2 \\ &\tilde{j}(Z, \tau) = \mathrm{i}\Psi_{E_0}(Z, \tau) \frac{\overleftrightarrow{\partial}}{\partial Z} \Psi_{E_0}^*(Z, \tau) \\ &\Psi_{E_0}(Z, \tau) = \beta^{1/2} \int_{-\infty}^{+\infty} \mathrm{d}x \, \mathrm{e}^{-\mathrm{i}x\tau} P(x)(a_x^2 + b_x^2)^{-1} \\ &\qquad \times \{\mathrm{Ai}(-\xi)\theta(-Z) + [a_x \, \mathrm{Ai}(-\xi) + b_x \, \mathrm{Bi}(-\xi)]\theta(Z)\} \qquad \xi = Z\beta^{-1} + \beta^2 x. \end{aligned} \tag{58}$$

From (52), (55) and (57) the following representations may be derived:

$$P(x)=\beta^{3/2}[e^{-\beta\xi_0}G(\beta)+e^{\beta\xi_0}F(\beta)]$$

$$F(\beta)=\int_{\xi_0}^{\infty}d\xi\, e^{-\beta\xi}[a_x\,\mathrm{Ai}(-\xi)+b_x\,\mathrm{Bi}(-\xi)] \tag{59}$$

$$G(\beta)=\int_{-\infty}^{\xi_0}d\xi\, e^{\beta\xi}\,\mathrm{Ai}(-\xi)\qquad \xi_0=\beta^2x.$$

The quantities $F(\beta)$ and $G(\beta)$ being considered as functions of β at fixed values of ξ_0, a_x and b_x obey the first-order differential equations

$$\frac{dF}{d\beta}-\beta^2F=e^{-\beta\xi_0}[a_x\,\mathrm{Ai}'(-\xi_0)+b_x\,\mathrm{Bi}'(-\xi_0)]$$

$$-\beta\, e^{-\beta\xi_0}[a_x\,\mathrm{Ai}(-\xi_0)+b_x\,\mathrm{Bi}(-\xi_0)]$$

$$\frac{dG}{d\beta}+\beta^2G=e^{\beta\xi_0}[\mathrm{Ai}'(-\xi_0)+\beta\,\mathrm{Ai}(-\xi_0)]$$

from which the following representations can be inferred:

$$e^{\beta\xi_0}F(\beta)=-\mathrm{Ai}'(-\xi_0)\,Y_0^{(+)}(x)-\mathrm{Ai}(-\xi_0)[2\beta Y_0^{(+)}(x)-Y_1^{(+)}(x)]$$

$$e^{-\beta\xi_0}G(\beta)=e^{-(\beta^3/3)-\beta\xi_0}\int_{-\xi_0}^{\infty}d\xi\,\mathrm{Ai}(-\xi)+\mathrm{Ai}'(-\xi_0)\,Y_0^{(-)}(x)+\mathrm{Ai}(-\xi_0)\,Y_1^{(-)}(x)$$

$$Y_n^{(+)}(x)=\beta^{n+1}\int_1^{\infty}dt\,t^n\,e^{-\beta^3q_x(t)}$$

$$Y_n^{(-)}(x)=\beta^{n+1}\int_0^{1}dt\,t^n\,e^{\beta^3q_x(t)}\qquad q_x(t)=\tfrac{1}{3}(t^3-1)+x(t-1).$$

We shall perform the further calculations under the assumption that the inequality

$$\beta\gg1 \tag{60}$$

is satisfied. The existence of the large parameter (β) permits us to use effectively the stationary phase method (Erdelyi 1962) and the method of integration by parts. The corresponding calculations are quite elementary though rather bulky. So we omit the intermediate calculations and write out only the final formula:

$$\begin{aligned}
P(x)&=\beta^{3/2}(1-(-x)^{-1/2})\exp[\beta^3(x+\tfrac{1}{3})] \qquad \text{at} \qquad x\leqslant-1-(1/\beta)\\
&=\frac{1}{4\sqrt{\pi}}\left\{2+\frac{2-x'}{2+x'}\exp\left[-\beta^3q_x\left(1+\frac{1}{\beta}\right)\right]\right\}\exp(-\tfrac{2}{3}\beta^3(-x)^{3/2})\\
&\quad-\tfrac{1}{4}x'\beta^{1/2}[1-\phi(a_-)]\exp(-\tfrac{2}{3}\beta^3(-x)^{3/2}+a_-^2)\\
&\quad\text{at } x=-1+\frac{x'}{\beta},\ |x'|\leqslant1,\ a_-=\frac{x'}{2}\beta^{1/2}\\
&=2\pi^{-1/2}\beta^{-6}(-x)^{-1/4}(1+x)^{-3}(1-(-x)^{1/2})\exp(-\tfrac{2}{3}\beta^3(-x)^{3/2})\\
&\quad\text{at } -1+\frac{1}{\beta}\leqslant x\leqslant-\frac{1}{\beta}\\
&=4\beta^{-7/2}(1+x)^{-3}[\mathrm{Ai}'(-\xi_0)+\beta\,\mathrm{Ai}(-\xi_0)]\qquad \text{at } x>-\frac{1}{\beta},\ \xi_0=\beta^2x.
\end{aligned} \tag{61}$$

With the aid of (56), (58) and (61) one can readily work out the quantities $\tilde{W}(\tau)$ and $\tilde{j}(Z, \tau)$. Note that the function $(a_E^2 + b_E^2)^{-1}$ has a pole at

$$E = E_0(1 + \tfrac{5}{16}\beta^{-6} \pm i\Gamma) \qquad \Gamma = \exp(-\tfrac{4}{3}\beta^3). \tag{62}$$

The formula (62) gives the energy and the damping of the quasistationary level arising in the potential well (14) under the influence of an electric field. Due to the existence of the pole the function $(a_x^2 + b_x^2)^{-1}$ involved in the integrands of the expressions (56) and (58) has the maximum at $x = -1 - \frac{5}{16}\beta^{-6} \equiv x_0$. The width of the maximum Δ defined by the equality

$$(a_x^2 + b_x^2)^{-1}|_{x = x_0 \pm \Delta} = \tfrac{1}{2} \max(a_x^2 + b_x^2)^{-1} = 2\Gamma^{-2}$$

coincides with the dimensionless damping: $\Delta = \Gamma$. In the region of β (60) the height of the maximum is rather considerable, but the width is small. Further, to obtain the numerical estimates we put:

$$|E_0| = 0.5 \text{ eV} \qquad \mathscr{E} = x\, 5 \times 10^5 \text{ V cm}^{-1}. \tag{63}$$

Then $\beta = 3.31\, x^{-1/3}$ and we have the following table.

x	β	$\frac{4}{3}\beta^3$	Γ
1	3.31	48.35	10^{-21}
2	2.63	24.18	3.2×10^{-11}
4	2.08	12.09	5.6×10^{-6}
6	1.82	8.06	3.2×10^{-4}

While calculating $\mathcal{M}_{E_0}(t)$ (56) let us first take into account the contribution to the integral made by a vicinity of the resonance point $x = x_0$:

$$\int_{x_0 - \delta}^{x_0 + \delta} dx\, e^{-ix\tau} |P(x)|^2 (a_x^2 + b_x^2)^{-1}$$

$$= e^{-ix_0\tau}\left[e^{-\Gamma\tau} - 2\pi^{-1}\Gamma\tau\left(\frac{\cos \delta\tau}{\delta\tau} - \frac{\pi}{2} + \mathrm{Si}(\delta\tau)\right)\right] \tag{64}$$

where $\mathrm{Si}(x) = \int_0^x dt\, t^{-1} \sin t$. In deriving this formula we have assumed that $\delta \sim \beta^{-4} \gg \Gamma$ and have taken into account the equality

$$P(x) = \frac{1}{2\sqrt{\pi}} \exp(-\tfrac{2}{3}\beta^3)$$

which is valid, according to (61), in the close vicinity of the resonance point. If only the contribution of the resonance point is allowed for, then for the ionisation probability we find

$$\tilde{W}(\tau) \simeq 2\Gamma \exp(-2\Gamma\tau). \tag{65}$$

By this formula the so-called exponential decay is described (Baz *et al* 1971). According to data given in the table, at $x = 1$ one act of ionisation occurs within the period of time $(\hbar/|E_0|)\Gamma^{-1} \sim 10^6$ s and thus $\tilde{W}(\tau) \sim 10^{-6}$ s^{-1}.

With the aid of (61) one can readily ascertain the contribution to the integral (56) made by the regions $x_0+\delta \leqslant x \leqslant -(1/\beta)$ and $x \leqslant x_0-\delta$ to be exponentially small. Thus, there remains only the contribution made by the vicinity of the resonance point $x=x_0$ and by the region $x \geqslant -(1/\beta)$. In view of (61) we obtain such an expression (assuming the inequality $\delta \gg \Gamma$ to be satisfied, we retain only the first term on the right-hand side of (64)):

$$\mathcal{M}_{E_0}(t) = \exp(-ix_0\tau - \Gamma\tau) + K\left(-\frac{1}{\beta}, \infty\right)$$
$$K(a,b) = 16\beta^{-7}\int_a^b dx\, e^{-ix\tau}(1+x)^{-6}[\mathrm{Ai}'(-\xi_0)+\beta\,\mathrm{Ai}(-\xi_0)]^2(a_x^2+b_x^2)^{-1}. \tag{66}$$

Estimate the quantity $\mathcal{M}_{E_0}(t)$ at large values of $\tau(\tau \geqslant \beta^{5/2})$. In this case the integrand in (66) is a high-frequency function, and so the main contribution to the integral is made by the integration limits and stationary points, the contribution of the latter being more considerable. Separate from (66) the integral $K(1/\beta, \infty)$ and consider the following factor of its integrand:

$$\frac{[\mathrm{Ai}'(-\xi_0)+\beta\,\mathrm{Ai}(-\xi_0)]^2}{a_x^2+b_x^2}$$
$$= \frac{\beta x^{1/2}}{2\pi}\,\frac{1-(2\cos 2\zeta/x^{1/2})+(1/x)(1+\sin 2\zeta)-\sin 2\zeta}{1-2(\cos 2\zeta/x^{1/2})+(2/x)(1+\sin 2\zeta)}$$
$$= \frac{\beta x^{1/2}}{2\pi}\sum_{s=-\infty}^{+\infty} D_s(x^{1/2})\,e^{i2s\zeta} \qquad \zeta = \tfrac{2}{3}\beta^3 x^{3/2}. \tag{67}$$

Here we have used the asymptotic form of the Airy function and performed the Fourier transformation. The expansion coefficients in (67) are of the form

$$D_s(x) = \begin{cases} \dfrac{i}{2}(x-i)^{-s+1} & s=1,2,\ldots \\ -\dfrac{i}{2}(x+i)^{-|s|+1} & s=-1,-2,\ldots \\ 1 & s=0. \end{cases}$$

Substituting (67) into the formula for $K(1/\beta, \infty)$, we obtain

$$K(1/\beta,\infty) = 16\pi^{-1}\beta^{-6}\sum_{s=-\infty}^{+\infty}\int_{\beta^{-1/2}}^{\infty} dx\,\frac{x^2}{(1+x^2)^6}\,D_s(x)\,e^{i\varphi_s(x)}$$
$$\varphi_s(x) = -x^2\tau + \tfrac{4}{3}s\beta^3 x^3. \tag{68}$$

In the region $\beta^{-1/2} \leqslant x < \infty$ the function $\varphi_s(x)$ has the stationary points $x = \tau/2s\beta^3 \equiv x_s$ for those values of $s=1,2,\ldots$ at which $x_s > \beta^{-1/2}$ (that is, at $\tau > 2s\beta^{5/2}$). Consider such values of τ for which the conditions $2\beta^{5/2} < \tau < 4\beta^{5/2}$ are fulfilled, provided that $|\tau - 2\beta^{5/2}|, |\tau - 4\beta^{5/2}| \gg 1$. Then there is the only stationary point $x = x_1 = \tau/2\beta^3$ on the path of integration. Using the stationary phase method, we arrive at

$$K(1/\beta,\infty) = -8\pi^{-1/2}\beta^{-6}\tau^{-1/2}x_1^2(1+x_1^2)^{-6}\exp\left[-\frac{i}{3}\tau\left(\frac{\tau}{2\beta^3}\right)^2 - i\frac{\pi}{4}\right]. \tag{69}$$

With the integral $K(-1/\beta, +1/\beta)$ there are no stationary points on the integration path, and so this integral may be neglected. From (58), (66) and (69) we now infer

$$\tilde{W}(\tau) = 2\Gamma\, e^{-2\Gamma\tau} + L_1 + L_2 \cos\left\{\tau\left[1+\frac{1}{3}\left(\frac{\tau}{2\beta^3}\right)^2\right]-\frac{\tau}{4}\right\}$$

$$L_1 = 192\pi^{-1}\beta^{-12}\tau^{-2}\frac{x_1^4(7x_1^2-1)}{(1+x_1^2)^{13}} \tag{70}$$

$$L_2 = -16\pi^{-1/2}\beta^{-6}\tau^{-1/2}\frac{x_1^2}{(1+x_1^2)^6}\, e^{-\Gamma\tau}.$$

The first term on the right-hand side of (70) allows for the contribution of the resonance point and gives the law of exponential decay. The third term is a result of interference between the resonance term and the non-exponential one in the matrix element (66). The values of all the three terms in (70) at $\tau = 3\beta^{5/2}$, $\beta = 3.31$ ($\Gamma\tau \ll 1$) are given below:

$$2\Gamma = 2\times 10^{-21} \qquad L_1 = 2\times 10^{-11} \qquad L_2 = 3.3\times 10^{-5}. \tag{71}$$

As is seen from (71), the contribution of the resonance point to the ionisation probability turns out to be very small. Physically, this means that in the region of β under study ($\beta \gg 1$) the direct tunnel transition of the particle from the quasistationary level in the potential well through the barrier is hardly probable. The main role is played by the processes indicated by Oleinik and Arepjev (1984a)—the quantum jumps of particles under the influence of the electric field from the quasistationary level to continuous spectrum states. These jumps do lead to the non-exponential law of decay.

Let us turn to calculating the probability flux density. We limit ourselves to consideration of the region $Z > 0$. The main contributions to the wavefunction $\Psi_{E_0}(Z, \tau)$ (58) are made by a vicinity of the resonance point $x = x_0$ and by the region $x \geqslant -1/\beta$. Denoting these contributions by $\Psi_1(Z, \tau)$ and $\Psi_2(Z, \tau)$, respectively, we have

$$\Psi_{E_0}(Z, \tau) = \Psi_1(Z, \tau) + \Psi_2(Z, \tau).$$

The probability flux density is defined as follows:

$$\tilde{j}(Z, \tau) = \tilde{j}_0(Z, \tau) + \tilde{j}_1(Z, \tau)$$

$$\tilde{j}_0(Z, \tau) = i\sum_{n=1,2}\Psi_n(Z, \tau)\frac{\overleftrightarrow{\partial}}{\partial Z}\Psi_n^*(Z, \tau) \tag{72}$$

$$\tilde{j}_1(Z, \tau) = i\Psi_2(Z, \tau)\frac{\overleftrightarrow{\partial}}{\partial Z}\Psi_1^*(Z, \tau) + \text{cc}.$$

The quantity $\tilde{j}_1$ will be referred to as the interference flux. The component $\Psi_1(Z, \tau)$ of the wavefunction is of the form

$$\Psi_1(Z, \tau) = (\pi\beta)^{1/2}\exp(-\tfrac{2}{3}\beta^3 + i\tau - \Gamma\tau)[\text{Bi}(-\tilde{\xi}) + i\,\text{Ai}(-\tilde{\xi})]$$
$$\tilde{\xi} = Z\beta^{-1} - \beta^2. \tag{73}$$

Making use of (73) we derive the law of exponential decay (see (70)):

$$i\Psi_1(Z, \tau)\frac{\overleftrightarrow{\partial}}{\partial Z}\Psi_1^*(Z, \tau) = 2\Gamma\, e^{-2\Gamma\tau}. \tag{74}$$

The last quantity turns out to be exponentially small; it may be neglected. By virtue of (58) and (73) the flux components $\tilde{j}_0$ and $\tilde{j}_1$ may be transformed into the form

$$\tilde{j}_0(Z,\tau) = 2\Psi_s(Z,\tau)\frac{\overleftrightarrow{\partial}}{\partial Z}\Psi_c(Z,\tau)$$

$$\tilde{j}_1(Z,\tau) = 2\pi^{1/2}\beta\, e^{-(2/3)}\beta^3\int_{-\beta^{-1}}^{\infty} dx\, \sin[(1+x)\tau]P(x)(a_x^2+b_x^2)^{-1} \tag{75}$$

$$\times[a_x\,\mathrm{Ai}(-\xi)+b_x\,\mathrm{Bi}(-\xi)]\frac{\overleftrightarrow{\partial}}{\partial Z}\mathrm{Bi}(-\tilde{\xi})$$

where the following notation is introduced:

$$\{\Psi_c(Z,\tau);\Psi_s(z,\tau)\}$$

$$= \beta^{1/2}\int_{-\beta^{-1}}^{\infty} dx\{\cos x\tau;\sin x\tau\}P(x)$$

$$\times(a_x^2+b_x^2)^{-1}[a_x\,\mathrm{Ai}(-\xi)+b_x\,\mathrm{Bi}(-\xi)]. \tag{76}$$

While calculating the probability flux density, we shall consider such distances Z that $\xi = Z\beta^{-1}+\beta^2 x \gg 1$. In this case we can take advantage of the asymptotic form of the Airy functions. Then in the integrand (76) there arise the rapidly oscillating factors of the type $\exp(i\varphi_{\pm}(x))$ where

$$\varphi_{\pm}(x) = x\tau \pm \tfrac{2}{3}(Z\beta^{-1}+\beta^2 x)^{3/2}.$$

The function $\varphi_+(x)$ has no stationary point, but the function $\varphi_-(x)$ has one at

$$x = \frac{1}{\beta^3}\left(\frac{\tau^2}{\beta^3}-Z\right) \equiv x_0^*.$$

Assume that $\tau \sim \beta^{5/2}$, $Z \sim \beta^2$. Then $x_0^* \leqslant 1/\beta$ and $\xi \sim \beta \gg 1$ and so the use of the asymptotic form of the Airy functions is valid. Using the stationary phase method in calculating the function (76), we obtain

$$\tilde{j}_0(Z,\tau) = 2\tau\beta^{-6}(a_{x_0^*}^2+b_{x_0^*}^2)^{-1}P^2(x_0^*)$$

$$P(x) = 4\beta^{-7/2}(1+x)^{-3}[\mathrm{Ai}'(-\xi_0)+\beta\,\mathrm{Ai}(-\xi_0)] \qquad \xi_0 = \beta^2 x. \tag{77}$$

The formula for the interference flux $\tilde{j}_1(Z,\tau)$ can be inferred analogously. Assuming that $\xi, \tilde{\xi} \gg 1$, we arrive at the relation

$$\tilde{i}_1(Z,\tau) = -2\tau^{1/2}\beta^{-3}\exp(-\tfrac{2}{3}\beta^3)P(x_0^*)(a_{x_0^*}^2+b_{x_0^*}^2)^{-1}$$

$$\times\left\{\left(\frac{\tilde{\xi}}{\xi}\right)^{1/4}\sin\left(\tfrac{2}{3}\tilde{\xi}^{3/2}+\frac{\pi}{4}\right)[a_{x_0^*}\sin(\tau+\varphi_-(x_0^*))-b_{x_0^*}\cos(\tau+\varphi_-(x_0^*))]\right.$$

$$\left.-\left(\frac{\tilde{\xi}}{\xi}\right)^{-1/4}\cos\left(\tfrac{2}{3}\tilde{\xi}^{3/2}+\frac{\pi}{4}\right)[a_{x_0^*}\cos(\tau+\varphi_-(x_0^*))+b_{x_0^*}\sin(\tau+\varphi_-(x_0^*))]\right\}. \tag{78}$$

The numerical estimate for the probability flux density is given at $\tau = 3\beta^{5/2}$, $\beta = 3.31$. According to (77) the maximum of the probability flux is reached at $x_0^* = 0$, that is at $Z = 9\beta^2$ (provided $\xi, \tilde{\xi} \gg 1$). So we have

$$\tilde{j}_0 \approx 10^{-5} \qquad \tilde{j}_1 \sim 10^{-14}. \tag{79}$$

From comparing (79) with (70) and (71) it is seen that the probability flux density turns out to be much greater in magnitude than the transition probability in unit time. It should be pointed out that in the calculations above we have used no expansions in perturbation.

As was explained in § 2, in the experiments on particle scattering the particle fluxes are registered. Therefore, it is natural to describe quantum processes in terms of the probability fluxes. Attention should be drawn to the following point. The calculation given above of the ionisation probability flux confirms the conclusion made earlier by virtue of the ionisation probability in unit time; namely, quantum jumps of a particle from the quasistationary states to those of the continuous spectrum are the main mechanism of ionisation at $\beta \gg 1$. However, the quantitative characteristics of the ionisation process obtained as a result of both calculations turn out to be quite different.

5. Quantum processes and vacuum vibrations

According to the conventional theory of quantum fields one of the main factors influencing physical properties and the behaviour of a microsystem is the continuous interaction of real particles with the vacuum 'as with such a type of physical media in which these particles move' (Bogoliubov and Shirkov 1976). Any real experimentally observed processes are considered to be accompanied by non-observable virtual processes due to the interaction of the microsystem with vacuum vibrations.

Let us trace the way of introducing the main concepts of quantum field theory by using as an example the simplest model—the electron-positron field interacting with an external field $A_{\text{ext}}(x)$ that acts only during the time interval (T_0, T_1). Denote by $\{\varphi_n^{(\pm)}(x)\}$ the complete orthonormalised set of functions obeying the Dirac equation for a free electron (n is the quantum number; the signs + and − refer to the positive-frequency and negative-frequency states, respectively; $x = (t, \boldsymbol{r})$). By the superposition principle an arbitrary solution of Dirac's equation may be represented as follows:

$$\sum_n [a_n \varphi_n^{(+)}(x) + b_n^* \varphi_n^{(-)}(x)] \equiv \Psi_0(x)$$

a_n and b_n being the constant coefficients. The transition to the second quantisation theory consists of replacing the coefficients a_n and b_n by the operators of the second quantization $\hat{a}_n$ and $\hat{b}_n$ to satisfy the ordinary anticommutation relations and to act in some space of state vectors. The quantity

$$\hat{\Psi}_0(x) = \sum_n [\hat{a}_n \varphi_n^{(+)}(x) + \hat{b}_n^+ \varphi_n^{(-)}(x)] \tag{80}$$

obtained in this way is called the free electron-positron field operator in the Heisenberg picture. The space of state vectors may be generated by operating the creation operators a_n^+ and b_n^+ on the vacuum ket vector $|0\rangle$ defined by the equalities $a_n|0\rangle = b_n|0\rangle = 0$ at any n. Here and below, the sign $\hat{}$ above the operators is omitted. These relationships together with the conjugate ones $\langle 0|a_n^+ = \langle 0|b_n^+ = 0$ and the normalisation condition $\langle 0|0\rangle = 1$ define the vacuum of the free electron-positron field.

To clarify the concept of a virtual particle, let us turn to the field operator $\Psi_0(x)$. The second quantisation operators a_n and b_n^+ involved in (80) may correspond both to the real observable particles and to the virtual non-observable ones depending on the state of the electron-positron field. For example, if the state is described by the

ket vector

$$|\phi_i\rangle = a_i^+|0\rangle \tag{81}$$

then the wavefunction $\varphi_n^{(+)}(x)$ at $n=i$ in (80) describes the real particle state and all the rest of wavefunctions should be attributed to the virtual states. Thus, we arrive at the notion of the virtual state as a vacancy (an empty state) in the space of one-particle states which may be occupied by a real physical particle (Oleinik 1986a). As is seen from (80), the virtual states (or particles) as well as the real particles are described by the same quantum numbers; the wavefunctions of real and virtual particles are included in (80) on equal grounds. The physical medium consisting of virtual particles will be called the vacuum background. An empty lecture hall with a great number of desks may serve as an image of the vacuum background. The empty seats at the desks correspond to the vacancies in the electron-positron field (the virtual particles) and the student at the desk is like a real particle that occupies the corresponding vacancy. Thus, the vacuum background is a set of vacancies which are described by certain wavefunctions and can be occupied by real particles.

Consider now the time evolution of the electron-positron field under the influence of external field. The field operator $\Psi(x)$ corresponding to particles in the external field $A_{\text{ext}}(x)$ is to satisfy the Dirac equation in this field and the initial condition

$$\Psi(x) = \Psi_0(x) \qquad \text{at } t = T_0. \tag{82}$$

From this we can infer the representation

$$\Psi(x) = \sum_n [a_n \psi_n^{(+)}(x) + b_n^+ \psi_n^{(-)}(x)] \tag{83}$$

where $\psi_n^{(\pm)}(x)$ are the solutions of Dirac's equation in the field $A_{\text{ext}}(x)$ obeying the initial condition

$$\psi_n^{(\pm)}(x) = \varphi_n^{(\pm)}(x) \qquad \text{at } t = T_0. \tag{84}$$

From (80) and (83) it is seen that the time evolution $\Psi_0(x) \to \Psi(x)$ may be treated as a 'dressing' of the 'bare' states $\varphi_n^{(\pm)}(x)$, both real and virtual, in the external field. The states of both real and virtual 'dressed' particles in the external field at a moment of time t $(t > T_0)$ are described by the wavefunctions $\psi_n^{(\pm)}(x)$ distinct from the free wavefunctions $\varphi_n^{(\pm)}(x)$. This means that the time evolution of the system in external field leads to the deformation of the vacuum background. The virtual particles existing in vacuum before and after the switching on of a perturbation differ from each other by their properties. If we recall our visual comparison of the vacuum background with the empty lecture hall, we may say that under the influence of an applied field the empty seats at desks are deformed. The point is that the processes of 'dressing' of virtual states take place irrespective of whether real particles are present in the system under study or not.

The processes responsible for the vacuum background distortions which inevitably occur in any microsystem subjected to a perturbation may be accompanied by the creation and annihilation of real electron-positron pairs. In these processes the virtual states are converted into the real ones and vice versa. Let us consider them in more detail following the paper by Oleinik (1985b). Introduce the subspaces $M_0^{(+)}$ and $M_0^{(-)}$ which are orthogonal to each other and are formed by linear combinations of the functions $\varphi_n^{(+)}$ and $\varphi_n^{(-)}$, respectively. These subspaces describe the upper and lower continua. The lower continuum is often called the Dirac sea of 'bare' particles. The

time evolution of one-particle states $\varphi_n^{(\pm)}(x) \to \psi_n^{(\pm)}(x)$ caused by an external field leads to the evolution of subspaces $M_0^{(\pm)}$:

$$M_0^{(\pm)} \to M^{(\pm)}(t).$$

Here $M^{(+)}(t)$ and $M^{(-)}(t)$ are the subspaces whose basis vectors are the functions $\psi_n^{(+)}(x)$ and $\psi_n^{(-)}(x)$, respectively. The subspace $M^{(-)}(t)$ describes at the moment t the Dirac sea of particles 'dressed' in the external field. It is obvious that the creation of an electron–positron pair in the external field should be connected with the formation of a hole in Dirac's sea of 'dressed' rather than 'bare' particles. To obtain the criterion for real pair creation, let us introduce the current density operator

$$j_\mu(x) = \tfrac{1}{2}[\bar{\Psi}(x)\gamma_\mu\Psi(x) - \Psi(x)\gamma_\mu^T\bar{\Psi}(x)].$$

The mean value of this operator in the vacuum state is given by

$$\langle 0|j_\mu(x)|0\rangle = \sum_n \bar{\psi}_n^{(-)}(x)\gamma_\mu\psi_n^{(-)}(x) - \sum_n \bar{\varphi}_n^{(-)}(x)\gamma_\mu\varphi_n^{(-)}(x) \equiv (\rho_{\text{vac}}(x), \boldsymbol{j}_{\text{vac}}(x)). \tag{85}$$

The creation of a hole in the subspace $M^{(-)}(t)$, that is in Dirac's sea of 'dressed' particles, results in the change of the particle density $\rho_{\text{vac}}(x)$ therein. Consequently, the condition for the real pair creation at a point x should be expressed by

$$\frac{\partial\rho_{\text{vac}}(x)}{\partial t} \neq 0 \tag{86}$$

which is equivalent, by the continuity equation, to

$$\operatorname{div}\boldsymbol{j}_{\text{vac}}(x) \neq 0. \tag{87}$$

The vacuum current $j_{\text{vac},\mu}(x)$ (85) may be represented in the form

$$\begin{aligned} j_{\text{vac},\mu}(x) &= J_\mu^{(+)}(x) + J_\mu^{(-)}(x) \\ J_\mu^{(\pm)}(x) &= \mp\frac{1}{2}\sum_n \bar{\psi}_n^{(\pm)}(x)\gamma_\mu\psi_n^{(\pm)}(x). \end{aligned} \tag{88}$$

It is natural to treat the quantities $J_\mu^{(\pm)}(x)$ as the currents induced by applied field in the subspaces $M^{(\pm)}(t)$. One can show that if the initial condition (84) is imposed on the functions $\psi_n^{(\pm)}$ at a finite moment of time T_0, the following equality is fulfilled:

$$J_\mu^{(+)}(x) = J_\mu^{(-)}(x). \tag{89}$$

It follows from (89) that the appearance of the vacuum current source in $M^{(-)}(t)$ inevitably entails the formation of the current source in $M^{(+)}(t)$. Therefore the creation of real pairs may be associated with the appearance of sources of the vectors $\boldsymbol{J}^{(+)}(x)$ and $\boldsymbol{J}^{(-)}(x)$. One can readily obtain the following formulae for the changes in the electric charge δQ and in the number of real particles $\delta\mathcal{N}$ in volume $\mathrm{d}\boldsymbol{r}$ in unit time:

$$\begin{aligned} \delta Q &= (|e|\operatorname{div}\boldsymbol{J}^{(-)}(x) - |e|\operatorname{div}\boldsymbol{J}^{(+)}(x))\,\mathrm{d}\boldsymbol{r} = 0 \\ \delta\mathcal{N} &= (\operatorname{div}\boldsymbol{J}^{(-)}(x) + \operatorname{div}\boldsymbol{J}^{(+)}(x))\,\mathrm{d}\boldsymbol{r} = \operatorname{div}\boldsymbol{J}_{\text{vac}}(x)\,\mathrm{d}\boldsymbol{r}. \end{aligned}$$

As is seen from these expressions, the interpretation proposed above does not lead to inconsistencies.

The validity of the criterion (87) for the creation and annihilation of a real electron–positron pair is also confirmed by simple qualitative considerations (Oleinik 1984, 1985b). Pair creation is tunnelling of the second type by which the particle with

negative energy is knocked out of the potential well. The role of the latter is played by the Dirac sea. The condition for tunnelling of the second type is given by

$$\operatorname{div} \boldsymbol{j}(\boldsymbol{r}, t) \neq 0 \tag{90}$$

where $\boldsymbol{j}(\boldsymbol{r}, t)$ is the probability flux density. If the field is in vacuum state, the only physical quantities we have for describing such a state are the quantities of the type $\langle 0|A|0\rangle$ where A is the operator of physical quantity. Therefore the condition for the creation or annihilation of the pair should be expressed by the inequality (90) in which $\boldsymbol{j}(\boldsymbol{r}, t)$ is replaced by $\boldsymbol{j}_{\text{vac}}(x)$.

It is obvious that if the transition $M_0^{(-)} \to M^{(-)}(t)$ consists only in distorting the lower continuum as a whole and is not accompanied by the appearance of sources or sinks of the vector $\boldsymbol{J}^{(-)}(x)$ in it, then the creation or annihilation of a real pair does not occur. In this case during time evolution of a microsystem the free virtual states $\varphi_n^{(\pm)}(x)$ are transformed into the 'dressed' virtual states $\psi_n^{(\pm)}(x)$. If, for example, the initial state is described by the ket vector (81), the wavefunction $\psi_i^{(+)}(x)$ describes the state of the real 'dressed' particle and all the rest of the $\psi_n^{(\pm)}(x)$ functions involved in (83) refer to 'dressed' virtual states.

Consider in greater detail the process of vacuum deformation. At $t \geqslant T_1$ the functions $\psi_n^{(\pm)}(x)$ may be expanded in free wavefunctions:

$$\psi_n^{(\pm)}(x) = \sum_m [\alpha_{nm}^{(\pm)} \varphi_m^{(+)}(x) + \beta_{nm}^{(\pm)} \varphi_m^{(-)}(x)] \equiv \varphi_n'^{(\pm)}(x) \tag{91}$$

where $\alpha_{nm}^{(\pm)}$ and $\beta_{nm}^{(\pm)}$ are constant coefficients. By $\Psi_0'(x)$ we denote the operator $\Psi(x)$ at $t > T_1$. In virtue of (91) the operator $\Psi_0'(x)$ obeys the Dirac equation for free particles. Thus, the operators $\Psi_0(x)$ and $\Psi_0'(x)$ describe the fields of free particles, but these fields essentially differ from each other. In fact, from the equalities

$$\langle 0|\Psi_0(x) a_n^+|0\rangle = \varphi_n^{(+)}(x) \qquad \langle 0|\Psi_0'(x) a_n^+|0\rangle = \varphi_n'^{(+)}(x)$$

it is seen that the particles described by the field operator $\Psi_0(x)$ are in stationary states and the particle states described by $\Psi_0'(x)$ are essentially non-stationary. Denote by $\mathscr{H}(x) = \mathscr{H}_0(x) + \mathscr{H}_{\text{int}}(x)$ the one-particle Hamiltonian of the particle in the field $A_{\text{ext}}(x)$ consisting of the free Hamiltonian $\mathscr{H}_0(x)$ and interaction Hamiltonian $\mathscr{H}_{\text{int}}(x)$. The Hamiltonian operators of free particles and of particles in the external field are given by the formulae

$$\begin{aligned} H_0 &= \int \mathrm{d}\boldsymbol{r} : \Psi_0^+(x) \mathscr{H}_0(x) \Psi_0(x) : = \sum_n \varepsilon_n (a_n^+ a_n + b_n^+ b_n) \\ H(t) &= S^+(t) \int \mathrm{d}\boldsymbol{r} : \Psi_0^+(x) \mathscr{H}(x) \Psi_0(x) : S(t) \end{aligned} \tag{92}$$

where : : is the operator of normal ordering, $S(t)$ is the operator of time evolution of electrons and positrons under the action of the field $A_{\text{ext}}(x)$ and ε_n is the free particle energy. At $t > T_1$ the operator $H(t)$ may be represented in the form

$$H(t) = S^+(t) H_0 S(t) = \sum_n \varepsilon_n (a_n^+(t) a_n(t) + b_n^+(t) b_n(t)) \equiv H_0'$$

$$\{a_n(t); b_n^+(t)\} = \sum_k (a_k \{\alpha_{kn}^{(+)}; \beta_{kn}^{(+)}\} + b_k^+ \{\alpha_{kn}^{(-)}; \beta_{kn}^{(-)}\}).$$

The latter equalities are easily obtained with the aid of the relation $\Psi(x) = S^+(t) \Psi_0(x) S(t)$ and equalities (80), (83) and (91). The operators H_0 and H_0' are the

operators of the total energy of the free electron-positron field prior to its interaction with the external field and after this interaction, respectively. We calculate the mean value of the energy in the vacuum state $\langle 0|H(t)|0\rangle \equiv E_0(t)$ at moments $t = T_0$ and $t = T_1$:

$$E_0(T_0) = 0 \qquad E_0(T_1) = \sum_n \varepsilon_n \sum_k (|\alpha_{kn}^{(-)}|^2 + |\beta_{kn}^{(+)}|^2) > 0. \tag{93}$$

If there is no real pair creation, that is the condition

$$\operatorname{div} j_{\text{vac}}(x) = 0 \tag{94}$$

is fulfilled, the quantities $E_0(T_0)$ and $E_0(T_1)$ represent the energy of the vacuum background in the ground state and the energy of the deformed vacuum background. Thus, the quantity $E_0(T_1)$ is an energy extracted by virtual particles from the external field and stored in the vacuum background. With the change in intensity of the field $A_{\text{ext}}(x)$, we obtain the energy band formed by levels relating to the vacuum background.

From the above it follows that the vacuum background inevitably appears in any quantum system investigated within the theory of second quantisation. The actual use of this theory means that the virtual particles and the vacuum background as a medium which the real particles 'inhabit' are taken into consideration. The Hamiltonian operator $H(t)$ (see (92)) expressed through the field operators automatically allows for the influence of the vacuum background on real particles.

Note that if there is no Dirac sea in the system under study, the vacuum background state is degenerate. Indeed, if in (80), (83) and (91) the wavefunctions $\varphi_n^{(-)}(x)$ and $\psi_n^{(-)}(x)$ are neglected, one will obtain $E_0(T_0) = E_0(T_1)$, although $\Psi_0'(x) \neq \Psi_0(x)$, $H_0' \neq H_0$.

Within conventional scattering theory the quantity $|\alpha_{im}^{(+)}|^2$ (see (91)) is interpreted as the transition probability of a real particle from the state $\varphi_i^{(+)}(x)$ at the moment $t = T_0$ to the state $\varphi_m^{(+)}(x)$ by the moment $t = T_1$ under the action of a perturbation. Because of the vacuum background deformation this interpretation turns out to be wrong. Indeed, let the initial state be described by the ket vector (81) and, additionally, let the condition (94) be valid. Then, in accordance with what has been said above, the time evolution of the field occurs in such a way that at the moment $t = T_1$ the wavefunction $\varphi_i'^{(+)}(x)$ describes the state of the real 'dressed' electron and all the rest of the $\varphi_n'^{(\pm)}(x)$ functions involved in (83) refer to 'dressed' virtual states. But the wavefunction $\varphi_m^{(+)}(x)$ taken at $t = T_1$ may be represented, as is seen from (91), in the form of superposition of both real ($\varphi_i'^{(+)}(x)$) and virtual ($\varphi_n'^{(+)}(x)$ for $n \neq i$ and $\varphi_n'^{(-)}(x)$ for any n) 'dressed' states. From this it follows that the quantity $|\alpha_{im}^{(+)}|^2$ is the population of the state m relating to the superposition of both real and virtual 'dressed' states. Therefore, the quantity $\alpha_{im}^{(+)}$ cannot be interpreted as the probability amplitude of the real physical process. This conclusion may be also drawn from comparing the Klein mechanism of vacuum polarisation with the non-stationary mechanism of pair creation predicted by Oleinik and Arepjev (1984b).

It should be stressed that when determining by virtue of (80) the field operator expansion at the initial moment $t = T_0$, we thereby fix the corpuscular interpretation of quantum theory. Because of the vacuum background deformation the 'dressed' states which may correspond to observable states of particles are described after switching off the interaction not by the wavefunctions $\varphi_n^{(\pm)}(x)$ but by the wavepackets $\varphi_n^{(\pm)}(x)$ (91). It is evident that the use of an analyser resolving the final state of a particle into initial states $\varphi_n^{(\pm)}(x)$ makes no physical sense. The vacuum background may be treated as a scene on which the real quantum events take place and which is

continuously deformed by perturbation. The basis of one-particle states $\varphi_n^{(\pm)}(x)$ at the moment $t = T_0$ does not coincide with the basis $\varphi_n'^{(\pm)}(x)$ referred to the moment $t = T_1$; as a result $\Psi_0'(x) \neq \Psi_0(x)$. This means that the standpoint of an observer on the physical properties of quantum state continuously varies with time.

Due to the vacuum background deformation the transition probabilities in unit time lose their physical meaning. The physical information on quantum dynamics is contained in the mean values of physical quantities in terms of which the behaviour of microsystems may be completely described.

Note that the corpuscular interpretation of quantum theory is not definitive. We might require, for example, that, instead of (82), the condition $\Psi(x) = \Psi_0'(x)$ at $t = T_0$ be satisfied. Then we should obtain another corpuscular interpretation which would lead to other physical predictions with respect to real quantum processes. The answer to the question: what interpretation is appropriate to Nature? may be found only on the basis of experimental data. Obviously, the choice of interpretation is determined by the choice of the vacuum state at the initial moment. In this connection the question arises (Oleinik 1985b): what is the structure of the physical vacuum in the real world?

The conclusions drawn above are a direct consequence of physical principles underlying the quantum theory. The conventional interpretation of quantities like $\alpha_{im}^{(\pm)}$ (91) as the probability amplitudes for real quantum processes disagrees with the principles mentioned above because it does not allow for the inevitable deformation of vacuum background under the influence of perturbation.

The operator $\Psi_0'(x)$ may be represented in the form

$$\Psi_0'(x) = \sum_n [a_n' \varphi_n^{(+)}(x) + b_n'^{+} \varphi_n^{(-)}(x)]$$

where

$$a_n' = S^+(t) a_n S(t) \qquad b_n'^{+} = S^+(t) b_n^+ S(t) \qquad \text{at } t \geqslant T_1.$$

As a rule, in case of an arbitrary external field $A_{\text{ext}}(x)$ the conditions

$$\sum_{n,m} |\beta_{nm}^{(+)}|^2 < \infty \qquad \sum_{n,m} |\alpha_{nm}^{(-)}|^2 < \infty$$

are not fulfilled. This means that the operators a_n, b_n and a_n', b_n' give unitary non-equivalent representations of the canonical anticommutation relations (Barton 1965, Oleinik 1979, Grib *et al* 1980). Therefore the above may be formulated as follows. During the process of time evolution of the quantum system in an external field there occurs the transition of the system to the unitary non-equivalent representations of the canonical anticommutation relations. The physical reason for this transition is the excitation of the vacuum background under the action of the external field.

To conclude this section let us dwell on the conventional interpretation of the problem of electron-positron pair creation in an external field and discuss the true physical meaning of those quantities which are calculated within the theory adopted at present (see Oleinik 1985b).

According to (91) after switching off the external field, that is at $t > T_1$, in the expansion of the wavefunction $\psi_n^{(-)}(x)$ there appear the positive-frequency components. This fact is usually interpreted as pair creation. The quantity

$$\alpha_{nm}^{(-)} = (\varphi_m^{(+)}, \psi_n^{(-)})_t \tag{95}$$

is assumed to be the probability amplitude for creating the pair. Thus, from the point

of view of the transition amplitude theory the inequality

$$\alpha_{nm}^{(-)} \neq 0 \tag{96}$$

is the pair creation condition. The quantity

$$\mathcal{N} = \sum_{n,m} |\alpha_{nm}^{(-)}|^2 \tag{97}$$

is interpreted as the total number of real pairs created in the external field.

It is clear that the condition (96) is far from being equivalent to (87). Indeed, in the electric field $\mathcal{E} = \mathcal{E}(t)$ arbitrarily varying in time, $\alpha_{nm}^{(-)} \neq 0$, but $\operatorname{div} \boldsymbol{j}_{\text{vac}}(x) = 0$. Obviously, the quantity $\mathcal{N}$ represents the number of states in a set which is the intersection of subspaces $M^{(-)}(t)$ and $M_0^{(+)}$. It may serve as a measure of the vacuum background distortions, but bears no relation to real electron–positron pair creation. As was explained above, real pairs are formed only if the vacuum current sources appear in the subspace $M^{(-)}(t)$, that is if $\operatorname{div} \boldsymbol{J}^{(-)}(x) \neq 0$. If the latter inequality is not fulfilled, the 'positrons' whose number is given by (97) are 'frozen' in the Dirac sea. They remain part of the Dirac sea and cannot be experimentally observed.

In the lowest order of perturbation theory

$$\alpha_{nm}^{(-)} = -\mathrm{i} \int_{-\infty}^{+\infty} \mathrm{d}\boldsymbol{r} \bar{\varphi}_m^{(+)}(x) e \hat{A}_{\text{ext}}(x) \varphi_n^{(-)}(x) \tag{98}$$

at $T_0 \to -\infty$, $T_1 \to \infty$. If we take as an example the field $\boldsymbol{A}_{\text{ext}}(x) = -\mathcal{E}\omega^{-1} \sin \omega t$, then for this field $\operatorname{div} \boldsymbol{j}_{\text{vac}}(x) = 0$, but $\alpha_{nm}^{(-)} \neq 0$ at $\omega > 2m$. Consequently, the vacuum may absorb the energy of the applied field with the frequencies $\omega > 2m$ without creating real pairs. The applied field energy is expended in redistribution of the vacuum charges ($\boldsymbol{j}_{\text{vac}}(x) \neq 0$) and is stored in the form of the vacuum background deformations (Oleinik 1983). At $t > T_1$ the virtual particles are in non-stationary states containing the components with both positive and negative frequencies. This means that at $t > T_1$ the vacuum turns out to be in the excited state which differs considerably from the ground vacuum state at $t = T_0$ (see (93)). The specific structure of the excited vacuum is defined by the coefficients in expansion (91) and depends on vacuum pumping by an applied field.

6. Conclusion

It is shown on the simplest models that there are considerable differences between the probability fluxes Π and the transition probabilities in unit time W. The essence of these differences is that the probability flux is the spacetime characteristic of the quantum process and the transition probability in unit time is the momentum–energy characteristic of the same process. The main difference between Π and W consists of the fact that the fluxes allow for the interference between the probability amplitudes relating to transitions to various quantum states while in the transition probabilities there are no interference terms.

It is noted that with decreasing duration of the electric field pulse causing the bound-state decay the difference between the quantities Π and W increases. However, there are regions of the values of parameters characterising quantum processes in which the quantities mentioned above do not practically differ from each other. Perhaps

this is the reason why up to now, while evaluating the cross sections of scattering processes, the difference between Π and W has not been taken into account and the transition probabilities, instead of the probability fluxes, have been used.

It is shown that in the case of a weak electric field the exponential law of bound-state decay is not valid. The direct transition of a particle from the quasistationary level in the potential well through the potential barrier resulting in the exponential law of decay turns out to be extremely unlikely. The main role in the decay is played by the quantum jumps of a particle under the influence of an electric field from the quasi-stationary level to the continuous spectrum states. The quantitative difference between the quantities Π and W in the problem on bound-state decay in a constant electric field is very great.

The existence of a vacuum background for real microsystems formed by virtual, non-observable particles and its continuous deformation during the time evolution is an inevitable consequence of the principles underlying quantum theory. The vacuum background may be treated as a framework in which the real quantum events occur. Its deformation leads to the continuous change in the point of view of observer on the physical properties of quantum state. Due to this the probability amplitudes lose their physical meaning. The quantity W represents the rate of change with time of the total population of the energy levels; these levels, however, refer partly to the real states and partly to the virtual ones. For this reason the transition probabilities W cannot describe quantum processes registered in experiment. The use of quantities of type W for describing quantum transitions may lead to incorrect qualitative conclusions and serious quantitative errors as is the case in the problem of electron-positron pair creation.

The deformation of the vacuum background is responsible for the appearance of an energy band of the vacuum states. The difference between the excited vacuum and the vacuum in the ground state consists of the fact that the virtual particles relating to these vacua are in essentially different quantum states, the energy of the former exceeding in magnitude the energy of the latter. The specific corpuscular interpretation of quantum theory is determined by the choice of the vacuum background at the initial moment.

As is seen from the results of the present paper and of Oleinik (1985b), the difference between Π and W is of significant character and may be illustrated by the example of any quantum process. These quantities may coincide in some ranges of parameters characterising the quantum transitions. However, such coincidences appear to be of a purely accidental nature. So far as the transition probabilities in unit time are still used instead of the probability fluxes when calculating the cross sections for scattering processes, it is necessary to revise the theoretical predictions concerning some quantum transitions. One should expect that considerable differences between Π and W will be found not only for the tunnel effects (the bound-state decay under the action of a perturbation, electron-positron pair creation in external fields and so on), but also for such quantum processes as Compton scattering, Möller scattering of electrons and others occurring in strong electromagnetic fields (see Oleinik 1967). Recently, serious differences between Π and W were found and investigated for the Čerenkov effect (Oleinik 1988).

Because of the existence of considerable quantitative differences between the quantities Π and W it is possible, in principle, to choose the theory of quantum transitions appropriate to Nature on the basis of experimental data. The carrying out of such experiments and the detailed investigation of the differences between Π and

W would make more precise the concept of the physical vacuum and could lead to a better insight into many quantum processes.

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